On the possibility of observation of the future for movement in the field of black holes of different types

Yuri V. Pavlov

Abstract It is shown for a spherically symmetric black hole of general type that it is impossible to observe the infinite future of the Universe external to the hole during the finite proper time interval of the free fall. Quantitative evaluations of the effect of time dilatation for circular orbits around the Kerr black hole are obtained and it is shown that the effect is essential for ultrarelativistic energies of the rotating particle.

Keywords Black holes · Kerr metric · Circular orbits

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1 Introduction

It is well known an observer falling radially into a black hole will reach the horizon in finite proper time, but the coordinate time in the Schwarzschild coordinate system is infinite [1,2]. This leads to an illusion of the possibility for a falling in the black hole by a cosmonaut so as to observe the infinite future of the Universe external to the black hole (see, for example, [3,4]). The impossibility of such an observation is shown in [5,6]. Note that the impossibility of the observation of the infinite future for the radial falling to the Schwarzschild black hole in the four-dimensional space-time is evident from the properties of the Kruskal–Szekeres coordinate system [7,8].

The possibility of observing the infinite future when falling on the black hole is analyzed in Sect. 2 for spherically symmetrical black holes of general type: black holes with electrical charge, with nonzero cosmological constant, dirty black holes (those with nonzero stress energy outside of static horizons), and multidimensional black holes. In the general case, there are no explicit

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analytical expressions for Kruskal–Szekeres coordinates and the analysis of space-time properties in such cases is difficult even for the radial falling on a black hole. In this paper it is shown from the analysis of the null and time-like geodesics for spherically symmetric black holes of the general type that, if the proper time of the fall is finite, then the time interval of observation of events in the point of the beginning of the fall is also finite. Quantitative evaluations are given for the Schwarzschild black holes.

Another the possibility to observe the far future of the external Universe is also discussed in the literature, namely due to time dilatation near a black hole. So in [9], p. 92, it is stated "However, a more prudent astronaut who managed to get into the closest possible orbit around a rapidly spinning hole without falling into it would also have interesting experiences: space-time is so distorted there that his clock would run arbitrary slow and he could, therefore, in subjectively short period, view an immensely long future timespan in the external universe". In Sect. 3 of the paper quantitative evaluations for the time dilatation on the circular orbits around the rotating black hole are obtained and it is shown that the effect becomes essential for ultrarelativistic energies of the rotating object.

2 Observation of the future when falling on the spherically symmetric black holes

Consider a spherically symmetric black hole with the metric

$$ds^2 = A(r)\, c^2 dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega_{N-2}^2 \,, \tag{1}$$

where c is the light velocity, $d\Omega_{N-2}$ — the angle element in space-time of the dimension $N \ge 4$, $A(r)$ — a certain function of the radial coordinate r which is zero on the event horizon r_H of the black hole: $A(r_H) = 0$. For the Schwarzschild black hole it holds [10]

$$A(r) = 1 - \frac{r_g}{r}\,, \qquad r_g = \frac{2GM}{c^2}\,, \tag{2}$$

where G is the gravitational constant, M — the mass of the black hole. For an electrically charged nonrotating black hole in vacuum we have [11,12]

$$A(r) = 1 - \frac{r_g}{r} + \frac{q^2}{r^2}\,, \tag{3}$$

where q is the charge of the black hole. For nonrotating black holes with nonzero cosmological constant Λ, one has Kottler [13] solution

$$A(r) = 1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}\,. \tag{4}$$

For multidimensional nonrotating charged black holes [14] with a cosmological constant, one has

$$A(r) = 1 - \left(\frac{r_g}{r}\right)^{N-3} - \frac{2\Lambda r^2}{(N-1)(N-2)} + \frac{q^2}{r^{2(N-3)}}, \tag{5}$$

where

$$r_g \overset{\text{def}}{=} \left[\frac{2G_N M}{(N-3)c^2}\right]^{1/(N-3)}, \tag{6}$$

provided G_N — the N-dimensional gravitational constant is normalized so that the N-dimensional Newton law in non-relativistic approximation possesses the following form

$$F = G_N \frac{mM}{r^{N-2}}. \tag{7}$$

Equations for geodesics in metric (1) can be written as

$$A(r)\frac{dt}{d\tau} = \varepsilon, \qquad \frac{d\varphi}{cd\tau} = \frac{L}{r^2}, \tag{8}$$

$$\left(\frac{dr}{cd\tau}\right)^2 = \varepsilon^2 - A(r)\left(\kappa + \frac{L^2}{r^2}\right), \tag{9}$$

where $\kappa = 1$ is for timelike geodesics and $\kappa = 0$ is for the null geodesics. For a particle with the rest mass m, the parameter τ is the proper time, $\varepsilon mc^2 = \text{const}$ is its energy in the gravitational field (1); and $Lmc = \text{const}$ is the projection of the angular momentum on the axis orthogonal to the plane of movement in the four-dimensional case.

From Eqs. (8), (9) for the intervals of the coordinate time $t_f - t_0$ and proper $\Delta\tau$ time of movement of the particle from the point with the radial coordinate r_0 to the point with coordinate $r_f < r_0$, one has

$$t_f - t_0 = \frac{1}{c}\int_{r_f}^{r_0} \frac{dr}{A(r)\sqrt{1 - \frac{A(r)}{\varepsilon^2}\left(\kappa + \frac{L^2}{r^2}\right)}}, \tag{10}$$

$$\Delta\tau = \frac{1}{c}\int_{r_f}^{r_0} \frac{dr}{\sqrt{\varepsilon^2 - A(r)\left(1 + \frac{L^2}{r^2}\right)}}. \tag{11}$$

As one can see from (10), the smallest coordinate time of movement is realized for photons with zero angular momentum. It is equal to

$$t_f - t_s = \frac{1}{c}\int_{r_f}^{r_0} \frac{dr}{A(r)}, \tag{12}$$

where t_s is the starting time for radial movement of the photon from the point r_0.

Subtracting (12) from (10) for $\kappa = 1$, one finds an answer to the question: how much later are the events in points with the same value of the radial coordinate as in the beginning of the fall which can be observed by the observer falling up to the point r_f?

$$t_s - t_0 = \frac{1}{c} \int_{r_f}^{r_0} \frac{\frac{1}{\varepsilon^2}\left(1+\frac{L^2}{r^2}\right) dr}{\sqrt{1-\frac{A(r)}{\varepsilon^2}\left(1+\frac{L^2}{r^2}\right)}\left[1+\sqrt{1-\frac{A(r)}{\varepsilon^2}\left(1+\frac{L^2}{r^2}\right)}\right]} . \tag{13}$$

From (11) and (13), one arrives at the following conclusion: if the proper time interval is finite, then the time interval of observation of the future events at the point of the beginning of the fall in the process of falling is also finite.

This conclusion is generalized for the case of movement of the charged particle. For the metric (5) this can be obtained by the transformation $\varepsilon \to \varepsilon - (qQ/r^{N-3})$, where Q is the specific charge of the moving particle in Eqs. (8)–(11), (13). If the energy, angular momentum and the charge of the particle are such that the proper time of the fall on the black hole is finite, then the observation of the infinite future of the external Universe is impossible.

In Fig. 1 the results of calculations of the ratio of the possible time of observation of the future at the point of the fall to the proper time interval for the observer radially falling in the Schwarzschild black hole are given. As we

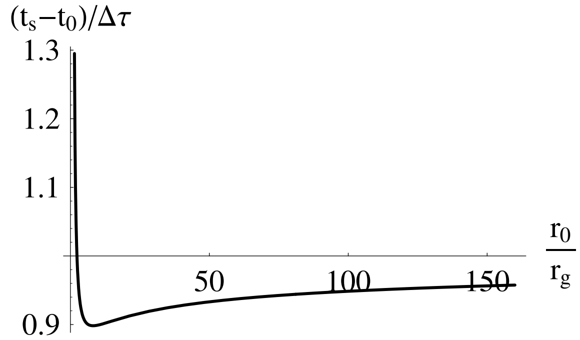

Fig. 1 The ratio $(t_s - t_0)/\Delta\tau$ for the observer falling from rest at the point r_0 to the horizon of the Schwarzschild black hole.

can see from (11), (13) for the Schwarzschild black hole (see explicit formulas (6), (9) from [5]) the following asymptotic behaviour holds $(t_s - t_0)/\Delta\tau \to 1$ for $r_0 \to \infty$. If the fall begins from rest at the point close to the event horizon, then

$$\frac{t_s - t_0}{\Delta\tau} \sim \sqrt{\frac{r_g}{r_0 - r_g}} \log 2 \to \infty, \qquad r_0 \to r_g . \tag{14}$$

But in this case $t_s - t_0 \approx (r_g/c)\,2\log 2$, i.e., the possible time interval of the observation of the future is small.

For any nonradial fall of the nonrelativistic particle on the Schwarzschild black hole it is also impossible to have a large interval of the future time which can be seen from Fig. 2:

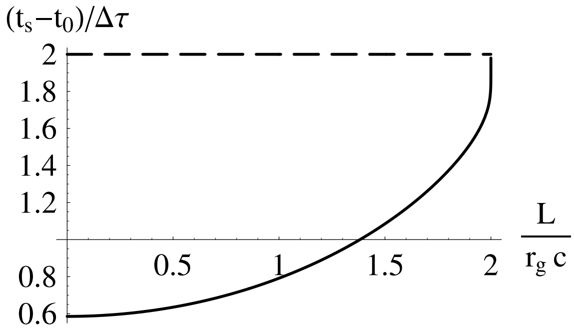

Fig. 2 The dependence $(t_s - t_0)/\Delta\tau$ on the angular momentum of the particle with $\varepsilon = 1$, $r_0 = 3r_g$ falling to the horizon of the Schwarzschild black hole.

3 Time dilatation on circular orbits of the Kerr black hole

Kerr's metric [15] of the rotating black hole in Boyer-Lindquist [16] coordinates has the form

$$ds^2 = dt^2 - \frac{2Mr\,(dt - a\sin^2\theta\,d\varphi)^2}{r^2 + a^2\cos^2\theta} - (a^2\cos^2\theta + r^2)\left(\frac{dr^2}{\Delta} + d\theta^2\right) - (r^2 + a^2)\sin^2\theta\,d\varphi^2, \tag{15}$$

where

$$\Delta = r^2 - 2Mr + a^2, \tag{16}$$

M is the mass of the black hole, aM — its angular momentum. Here we use the units: $c = G = 1$. For $a = 0$, the metric (15) describes a nonrotating black hole in Schwarzschild coordinates. The event horizon of the Kerr's black hole corresponds to the radial coordinate

$$r = r_H \equiv M + \sqrt{M^2 - a^2}\,. \tag{17}$$

Equatorial ($\theta = \pi/2$) geodesics in Kerr's metric (15) are defined by the equations (see [17], Sect. 61):

$$\frac{dt}{d\tau} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\varepsilon - \frac{2Ma}{r}L\right], \tag{18}$$

$$\frac{d\varphi}{d\tau} = \frac{1}{\Delta} \left[\frac{2Ma}{r}\, \varepsilon + \left(1 - \frac{2M}{r} \right) L \right], \tag{19}$$

$$\left(\frac{dr}{d\tau} \right)^2 = \varepsilon^2 + \frac{2M}{r^3}\, (a\varepsilon - L)^2 + \frac{a^2 \varepsilon^2 - L^2}{r^2} - \frac{\Delta}{r^2} \kappa, \tag{20}$$

where $\varepsilon m = \mathrm{const}$ is the energy of the particle with the rest mass m in the gravitational field (15); $Lm = \mathrm{const}$ is the projection of the angular momentum of the particle on the rotation axis of the black hole.

Let us define the effective potential of the particle in the field of the black hole by

$$V_{\mathrm{eff}} = -\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 . \tag{21}$$

Then $d^2 r / d\tau^2 = -dV_{\mathrm{eff}}/dr$ and the necessary conditions for the existence of circular orbits in equatorial plane are

$$V_{\mathrm{eff}} = 0, \qquad \frac{dV_{\mathrm{eff}}}{dr} = 0\,. \tag{22}$$

It is sufficient for the existence of stable circular orbits that

$$V_{\mathrm{eff}} = 0, \qquad \frac{dV_{\mathrm{eff}}}{dr} = 0, \qquad \frac{d^2 V_{\mathrm{eff}}}{dr^2} > 0\,. \tag{23}$$

The solutions of Eqs. (22) can be written in the form [18]

$$\varepsilon = \frac{x^{3/2} - 2\sqrt{x} \pm A}{\sqrt{x\,(x^2 - 3x \pm 2A\sqrt{x}\,)}}\,, \tag{24}$$

$$l = \pm \frac{x^2 \mp 2A\sqrt{x} + A^2}{\sqrt{x\,(x^2 - 3x \pm 2A\sqrt{x}\,)}}\,, \tag{25}$$

where the upper sign corresponds to the direct orbits (i.e., the orbital angular momentum of a particle is parallel to the angular momentum of the black hole), the lower sign corresponds to retrograde orbits,

$$x = \frac{r}{M}\,, \qquad A = \frac{a}{M}\,, \qquad l = \frac{L}{M}\,. \tag{26}$$

The circular orbits exist from $r = \infty$ up to minimal value corresponding to the photon circular orbit r_{ph} defined by the roots of the denominator (24), (25) equal to [18]

$$r_{\mathrm{ph}}^{\pm} = 2M \left[1 + \cos \left(\frac{2}{3} \arccos(\mp A) \right) \right] . \tag{27}$$

The minimal radius of the stable circular orbit is equal to [18]

$$x_{\mathrm{ms}}^{\pm} = 3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\,, \tag{28}$$

where

$$Z_1 = 1 + \left(1-A^2\right)^{1/3} \left[(1+A)^{1/3} + (1-A)^{1/3}\right], \qquad Z_2 = \sqrt{3A^2 + Z_1^2}\,. \tag{29}$$

The specific energy of the particle on such a limiting stable orbit is $\varepsilon = \sqrt{1-(2/3x_{\rm ms})}$.

The minimal radius of the bounded orbit (i.e., orbit with $\varepsilon < 1$) is obtained for $\varepsilon = 1$ (the particle is nonrelativistic at infinity) and is equal to [18]:

$$x_{\rm mb}^{\pm} = 2\left(1 + \sqrt{1 \mp A}\right) \mp A\,. \tag{30}$$

In this case, we have

$$l_{\rm mb}^{\pm} = \pm 2\left(1 + \sqrt{1 \mp A}\right). \tag{31}$$

This orbit is nonstable.

From (18), (24), (25) one obtains

$$\frac{dt}{d\tau} = \frac{x^{3/2} \pm A}{\sqrt{x^{3/2}(x^{3/2} - 3\sqrt{x} \pm 2A)}}\,. \tag{32}$$

Due to the fact that t is the time of the observer resting at infinity from the black hole, τ is the proper time of the observer moving along the geodesics, one can consider this value to be "time dilatation" for the corresponding circular orbit.

For the Schwarzschild black hole ($A = 0$), the time dilatation on the limiting bounded circular orbit ($x_{\rm mb} = 4$) is equal to 2 (the limiting horizontal dashed-line on Fig. 2 corresponds to this value). On the minimal stable circular orbit ($x_{\rm ms} = 6$), the time dilatation is only $\sqrt{2}$. However, for circular orbits close to photon orbit $r_{\rm ph} = 3M$, from (24), (32) for $A = 0$, using the series expansion in ε^{-1} one gets

$$\frac{dt}{d\tau} = 3\varepsilon + \frac{1}{6\varepsilon} + O\left(\frac{1}{\varepsilon^3}\right), \quad \varepsilon \to \infty, \tag{33}$$

i.e., the time dilatation can be as large as possible.

Note that in Minkowski space-time one gets from the special relativity $dt/d\tau = \varepsilon$. So in the case of movement around the Schwarzschild black hole, the time dilatation for relativistic energies of the objects can be enlarged at most by a factor of three.

For circular orbits with $\varepsilon \to \infty$ around the rotating black holes formulas (24), (27), (32) give

$$\frac{dt}{d\tau} \sim \left[3 \pm \frac{2A}{2\cos\left(\frac{\arccos(\mp A)}{3}\right) \mp A}\right]\varepsilon. \tag{34}$$

For direct orbits close to the rapidly rotating black holes with $A \to 1$, $r_{\rm ph}^{+} \to M$, it holds

$$\frac{dt}{d\tau} \sim \left(\sqrt{\frac{6}{1-A}} + \frac{1}{3} + O(\sqrt{1-A}) \right) \varepsilon, \tag{35}$$

for retrograde orbits $r_{\rm ph}^{-} \to 4M$,

$$\frac{dt}{d\tau} \sim \left(\frac{7}{3} + O(1-A) \right) \varepsilon. \tag{36}$$

The time dilatation on the minimal stable circular orbit of the rapidly rotating black hole ($A \to 1$) can be obtained from (24), (28)–(32) and is equal, for direct orbit $x_{\rm ms}^{+} \to 1$,

$$\frac{dt}{d\tau}(x_{\rm ms}^{+}) = \frac{2^{\,4/3}}{\sqrt{3}\,(1-A)^{1/3}} \left[1 + O(\sqrt[3]{1-A}) \right], \tag{37}$$

for the retrograde orbit $x_{\rm ms}^{-} \approx 9$, $dt/d\tau \approx 13/(6\sqrt{3})$. For minimally bound direct circular orbit, $x_{\rm mb}^{+} \to 1$, we have

$$\frac{dt}{d\tau}(x_{\rm mb}^{+}) = \frac{2}{\sqrt{1-A}} \left[1 + O(\sqrt{1-A}) \right]. \tag{38}$$

For the retrograde minimally bound orbit: $x_{\rm mb}^{-} \approx 3 + 2\sqrt{2}$, $dt/d\tau \approx 3 - \sqrt{2}$.

Let us give the values of time dilatation on circular orbits for the black hole with the Thorne's limit for astrophysical black holes $A = 0.998$ (see [19]):

$$\frac{dt}{d\tau}(x_{\rm mb}^{+}) = 43.8, \quad \frac{dt}{d\tau}(x_{\rm ms}^{+}) = 10.8. \tag{39}$$

For particles with large specific energy rotating in the direction of rotation of the black hole for orbits close to circular photon orbit one has

$$A = 0.998 \quad \Rightarrow \frac{dt}{d\tau} \sim 55.12\,\varepsilon, \quad \varepsilon \to \infty. \tag{40}$$

So the relativistic effect of the time dilatation close to a rotating black hole may be 55 times larger!

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